FAST WAVELET ANALYSIS OF 3-D DIELECTRIC STRUCTURES USING SPARSE MATRIX TECHNIQUES

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Abstract. This paper presents an efficient integral formulation based on the theory of orthonormal wavelets for the analysis of open three-dimensional dielectric structures used in microwave and optical applications. In this approach, the fields and currents are represented by a two-dimensional multiresolution expansion in a transverse plane and a subdomain pulse expansion along the normal direction. The implementation of the method of moments is then combined with the highly efficient Fast Wavelet Algorithm (FWA). It is shown that the resulting moment matrices are very sparsely populated and easily render themselves to the sparse matrix techniques like the Bi-Conjugate Gradient method. Finally, to validate the formulation, a rectangular dielectric resonator is investigated.

I. Introduction

The full-wave analysis of open dielectric structures is a key factor in the successful design of submillimeter-wave and optical integrated circuit components. The two-dimensional dielectric geometries have been studied exhaustively during the past two decades, and numerous techniques have been either elaborately devised or simply extended to treat this type of geometries. Nonetheless, many of these methods are limited in scope and applicability to 2-D or extremely simple 3-D problems. An accurate study of complex three-dimensional dielectric structures requires rigorous numerical methods which do not suffer from excessive oversimplification of the problem. Such methods are based on the discretization of the geometry into a very fine mesh, where an appropriate numerical scheme for the solution of the related boundary-value problem is implemented [1]-[2]. Discretization in three dimensions usually leads to a prohibitive intensity in the computational part of the problem. The methods based on integral formulations enjoy the privilege that only finite regions of the geometry, and not the entire space, need to be discretized. Yet, the numerical implementation of the integral equation technique can easily turn into a formidable computational task and rule out its practicality in competition with the other numerically intensive approaches. The bottleneck is the full moment matrices which result from the use of the method of moments for the numerical solution of the integral equations. It has been shown that this major obstacle can be removed by using orthonormal wavelet expansions for the expansion of electromagnetic fields and currents. Over the past two years, several authors have demonstrated this possibility for a variety of two-dimensional electromagnetic problems [3]-[4]. In particular, reference [3] presents a mixed spectral/space-domain integral formulation for two-dimensional dielectric waveguide structures.

In this paper, the application of the multiresolution analysis theory to the solution of three-dimensional volume integral equations is demonstrated for the first time. We develop a space-domain integral formulation for open 3-D dielectric structures, and show that the use of wavelet expansions in a 3-D method of moments still leads to very sparsely populated moment matrices after performing a threshold procedure. Moreover, by exploiting a highly efficient recursive algorithm, called the Fast Wavelet Algorithm (FWA), the computation of a large portion of the moment integrals is reduced to discrete convolutions. To illustrate the efficiency of this approach, the problem of a rectangular dielectric resonator is considered. By examining the excited field due to a prescribed source, the resonant frequency of the dielectric resonator can be determined. Since the moment matrix is highly sparse, one can combine sparse storage techniques with a pre-conditioned Bi-Conjugate Gradient method to increase the computational efficiency to a large extent in view of both speed and memory space. An-

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other advantage of this approach is that the field distribution is also computed at the same time.

II. Integral Formulation

In this section, we develop a three-dimensional integral formulation for a general dielectric structure embedded in a planar multilayered background geometry with a known dyadic Green's function, as shown in Fig. 1. An equivalent volume polarization current $\boldsymbol{J_p}(\boldsymbol{r})$ is introduced in the following manner:

$$\boldsymbol{J_p}(\boldsymbol{r}) = jk_0 Y_0\, \delta\epsilon_r(\boldsymbol{r})\, \boldsymbol{E}(\boldsymbol{r})\, p_V(\boldsymbol{r}) \tag{1}$$

where $\boldsymbol{E}(\boldsymbol{r})$ is the electric field, k_0 and $Y_0 = 1/Z_0$ are the free-space propagation constant and characteristic admittance, respectively, $\delta\epsilon_r(\boldsymbol{r})$ is the index contrast of the dielectric region with respect to its cover medium, and $p_V(\boldsymbol{r})$ is the characteristic function of the volume V occupied by the dielectric. Then, having known the dyadic Green's function $\bar{\bar{\boldsymbol{G}}}_{\boldsymbol{e}}(\boldsymbol{r} \mid \boldsymbol{r}')$ of the background structure, one can express the electric field at any point in the following form:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \boldsymbol{E^i}(\boldsymbol{r}) \\ &- jk_0 Z_0 \int\!\!\int\!\!\int_V \bar{\bar{\boldsymbol{G}}}_{\boldsymbol{e}}(\boldsymbol{r} \mid \boldsymbol{r}') \,.\, \boldsymbol{J_p}(\boldsymbol{r}')dv' \end{aligned} \tag{2}$$

where $\boldsymbol{E^i}(\boldsymbol{r})$ is the incident electric field due to a prescribed source. By combining equations (1) and (2), the desired integral equation for the unknown polarization current is obtained. Note that the dyadic Green's function in this case consists of a Cauchy principal value plus a source dyadic due to the singularity at the source.

III. Two-Dimensional Multiresolution Analysis

This section presents a two-dimensional generalization of the concept of the multiresolution analysis (MRA). A detailed account of the theory of one-dimensional MRAs is given in reference [3], and here the emphasis is rather placed on the special features of 2-D multiresolution expansions, which will be used later in the moment method solution of our integral equation.

A two-dimensional multiresolution analysis is a ladder of successive nested approximation subspaces $\boldsymbol{V_m}$ of $\boldsymbol{L^2(R^2)}$. Each subspace $\boldsymbol{V_m}$ at a resolution of 2^{-m} is constructed from the Cartesian product of two one-dimensional MRAs at the same resolution, i.e., $\boldsymbol{V_m} = V_m \otimes V_m$ [5]. Then, $\boldsymbol{V_m}$ is spanned by the orthonormal basis $\Phi_{m,n_x,n_y}(x,y) = \phi_{m,n_x}(x)\phi_{m,n_y}(y), n_x, n_y \in Z$, where the one-dimensional functions are properly dilated and shifted versions of the scaling function $\phi(x)$ or $\phi(y)$, respectively. Each subspace $\boldsymbol{V_m}$ has an orthogonal complement in the next finer subspace $\boldsymbol{V_{m+1}}$, which is denoted by $\boldsymbol{W_m}$, and it can be shown that

$$\boldsymbol{W_m} = W_m \otimes V_m \,\oplus\, V_m \otimes W_m \,\oplus\, W_m \otimes W_m \tag{3}$$

where W_m is the one-dimensional complement space. Hence, $\boldsymbol{W_m}$ is spanned by an orthonormal basis consisting of three sets of wavelets:

$$\begin{aligned} \Psi^v_{m,n_x,n_y}(x,y) &= \psi_{m,n_x}(x)\phi_{m,n_y}(y), \\ \Psi^h_{m,n_x,n_y}(x,y) &= \phi_{m,n_x}(x)\psi_{m,n_y}(y), \\ \Psi^d_{m,n_x,n_y}(x,y) &= \psi_{m,n_x}(x)\psi_{m,n_y}(y), \end{aligned}$$

with $\psi(x)$ and $\psi(y)$ being the mother wavelet. These wavelets are called vertical, horizontal and diagonal, respectively, in that they represent the variations of the expanded function primarily along the corresponding directions.

Now one can express the approximation of an arbitrary square-integrable 2-D function $f(x,y)$ at a resolution of 2^{-m} by defining a projection operator $P_m(f)$ onto the subspace $\boldsymbol{V_m}$ in the following form:

$$P_m(f) = \sum_{n_x \in Z} \sum_{n_y \in Z} < f, \Phi_{m,n_x,n_y} > \Phi_{m,n_x,n_y}(x,y) \tag{4}$$

Then the improved approximation at the next finer resolution, 2^{-m-1}, is given by

$$\begin{aligned} P_{m+1}(f) &= P_m(f) + \\ &\sum_{i=v,h,d} \sum_{n_x \in Z} \sum_{n_y \in Z} < f, \Psi^i_{m,n_x,n_y} > \Psi^i_{m,n_x,n_y}(x,y) \end{aligned} \tag{5}$$

and the improvement can be continued to any arbitrary resolution by including the intermediate 2-D wavelets.

It is important to note that the definition of a multiresolution analysis requires that $\int_\infty^\infty \Phi(x,y)dxdy = 1$, and $\int_\infty^\infty \Psi^i(x,y)dxdy = 0$, $i = v, h, d$. Thus, in expanding a 2-D function which has a compact support over a given domain, say D, one can judiciously choose wavelets of types which best represent the discontinuities at certain parts of the boundary of this domain. For example, it would be efficient to place more diagonal wavelets at the corners of the region, and not at its center which is rather free of diagonal discontinuities. Such a selection leads to a significant economization of the expansion basis by discarding the redundant expansion functions. Fig. 2 shows typical locations where different types of basis functions are placed.

IV. Numerical Results

To implement the method of moments, the volume polarization current $\boldsymbol{J_p(r)}$ is expanded in a 2-D multiresolution expansion in the x-y plane and a sub-domain pulse basis along the z-axis in the following manner:

$$\boldsymbol{J_p(r)} = \sum_i \sum_p \boldsymbol{a_{ip}} \, F_i(x,y) \, p_p(z) \tag{6}$$

where $F_i(x,y)$ denotes a 2-D multiresolution basis function, either of the Φ- or Ψ-type, and $p_p(z)$ denotes a sub-domain pulse function. This expansion is used to discretize the integral equation, and then by applying a Galerkin testing procedure, the following linear system of matrix equations is obtained:

$$\left[\bar{\bar{\boldsymbol{K}}}\right] \cdot [\boldsymbol{a}] = [\boldsymbol{b}] \tag{7}$$

where $[\bar{\bar{\boldsymbol{K}}}]$ is the moment matrix, and $[\boldsymbol{a}]$ and $[\boldsymbol{b}]$ denote the amplitude vector and excitation vector, respectively.

Due to the very large number of expansion functions, the computation of the moment integrals can take a very long CPU time. The scaling function and mother wavelet do not have simple closed-form expressions, and in particular, those multiresolution functions which are truncated by the boundaries, decay quite slowly in the Fourier domain. However, some interesting inherent properties of the multiresolution analysis makes it possible to reduce this computation time drastically by using the Fast Wavelet Algorithm (FWA). This highly efficient recursive algorithm is based on the following two-scale properties:

$$\begin{aligned} \phi(x) &= \sqrt{2} \sum_n h_n \, \phi(2x-n) \\ \psi(x) &= \sqrt{2} \sum_n g_n \, \phi(2x-n) \end{aligned} \tag{8}$$

where $\{h_n\}$ and $\{g_n\}$ are discrete sequences characteristic of the MRA. It can be shown that having known the multiresolution expansion coefficients of a given function $f(x)$ with respect to the set $\phi_{m,n}(x)$, $n \in Z$ at a resolution of 2^{-m}, one can easily determine the coefficients with respect to both scaling functions and wavelets at all coarser resolutions by simple discrete convolutions. This scheme can easily be extended to more than one dimension. In particular, this work exploits a 4-D version of the FWA for the computation of the moment integrals.

As an example, in this paper we consider the problem of a rectangular dielectric resonator embedded in the free space, as shown in the inset of Table 1. To determine the resonant frequency of this structure, a prescribed source such as a plane wave is applied, and the excited field due to this source is computed using a pre-conditioned Bi-Conjugate Gradient (Bi-CG) method. By changing the polarization of the incident field, it is possible to study different excited modes of the structure. A typical set of resonator parameters are $\epsilon_{rg} = 20.0$, $a = 10mm$, $b = 8mm$, and $h = 5mm$, where a, b, and h are the dimensions of the resonator along the $x-$, $y-$, and z-axes, respectively. Table 1 shows the computed resonant frequencies of the dominant and the next two modes of this resonator. The results have been compared to those based on Marcatili's approximation [6], and a good agreement is observed. Here, we have adopted the cubic spline Battle-Lemarie multiresolution analysis, and $m_0 = 2$ has been taken as the initial resolution level. Only two resolution levels and 2-4 pulse functions along the normal direction are sufficient to yield very accurate results. The resulting moment matrix, as expected, is highly sparse in the sense that a very large number of its entries are quite small in magnitude when compared to the largest entry. Fig. 3 shows the structure of the moment matrix after applying a threshold of 1%. In this case, the expansion basis consists of 2 pulse functions and a total of 231 2-D multiresolution expansion functions, and the sparsity of the moment matrix is 99.16%. Using special sparse matrix techniques such as the row-indexed sparse storage scheme in combination with the a pre-conditioned Bi-CG method yields an extremely fast tool for field computations. The resonant frequency of the dielectric resonator is determined through searching for the prescribed source frequency for which the excited field inside the resonator reaches a maximum.

V. Conclusion

The application of 2-D multiresolution expansions to three-dimensional electromagnetic problems, in particular, open dielectric structures, has been demonstrated. It is expected that the combination of the MRA theory with the method of moments can provide a very powerful and efficient computational tool for the full-wave study of largescale and complex 3-D electromagnetic problems.

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Table I
Resonant Frequencies of Different Modes of a Rectangular Dielectric Resonator

Mode	f_{res} (This work)	f_{res} (Ref.[6])
TM^x_{111}	5.532 GHz	5.651 GHz
TM^y_{111}	6.012 GHz	6.104 GHz
TE^z_{111}	6.378 GHz	6.575 GHz

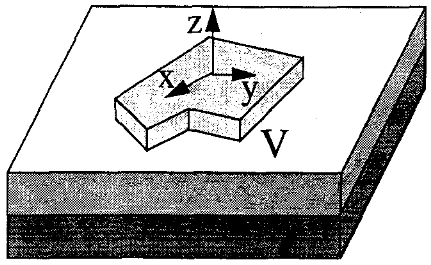

Fig.1. Geometry of the problem.

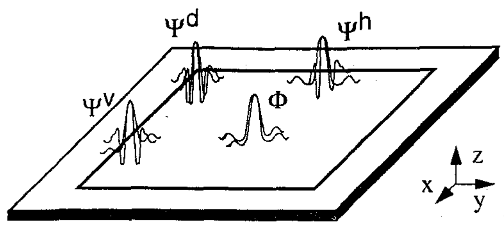

Fig.2. Selection of 2-D multiresolution expansions.

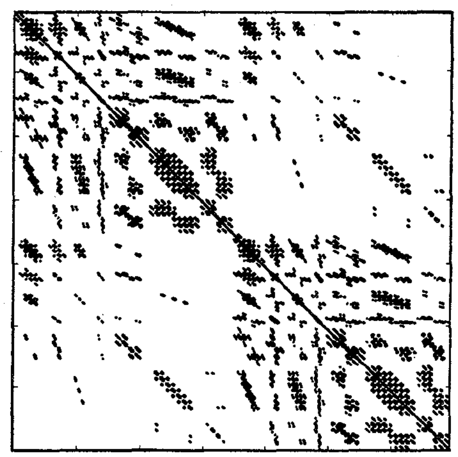

Fig 3. Structure of moment matrix after thresholding.